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Linear stability of perturbed Hamiltonian systems: theory and a case example

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Abstract

In this paper we present a general theory, based on a Lyapunov–Schmidt reduction, for the linearized stability of a perturbed Hamiltonian system with a number of symmetries. The reduction leads to an eigenvalue problem in a small subspace of the original system, whose eigenvalues and eigenvectors can be used to derive the ones of the perturbed original system. The Krein signature of the eigenvalues can also be obtained in this framework. The method is applied to the case example of the coupled nonlinear Schrödinger equations giving excellent agreement with the full numerical linear stability results.

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1. Introduction

Integrable Hamiltonian nonlinear systems of partial differential equations (PDEs) or differential difference equations (DDEs) have been the backbone on which soliton theory has been based [1]. On the other hand, it is by now well understood that in most of the physically relevant applications, from arrays of waveguides [2] and optical fibres [3] to Bose–Einstein condensates in traps [4] and Josephson junctions [5], the realistic physical models correspond to perturbed (non-integrable) variations of such models. In fact, oftentimes these variations may even be dissipative in nature, even though this will not be of interest to us here. Hence, it is naturally of interest to have a methodology such that, given the exact knowledge of a specific system (such as e.g., an integrable one), and of the nature of the (Hamiltonian) perturbation to it, conclusions can be drawn on the existence and stability of nonlinear waves in the perturbed system.

Our starting point will be a perturbed system of the general form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = JE'(u) \tag{1.1}$$

where J is an invertible skew-symmetric operator with bounded inverse and $E(u) = E_0(u) + \epsilon E_1(u)$, with $0 < \epsilon \ll 1$. Here $E(u)$ represents the total energy of the system. The underlying assumption is that the perturbation breaks some of the symmetries of the unperturbed problem. Skryabin [6] studied perturbations that broke one specific symmetry. A generalization of his results which also extends the results of [7] on existence and local stability analysis will be presented herein. These results will be applied to a specific problem, namely the stability of nonlinear waves in linearly coupled optical fibres. The latter are described by the coupled nonlinear Schrödinger (CNLS) equations of the form.

$$\mathrm{i}\partial_t q_j = -\frac{1}{2}\left(\partial_{xx}^2 q_j - q_j\right) - |q_j|^2 q_j - \frac{\epsilon}{2}(\Delta_2 q)_j \tag{1.2}$$

where $(\Delta_2 q)_j := q_{j+1} - 2q_j + q_{j-1}$ and $\mathrm{i} := \sqrt{-1}$. In this equation q_j represents the complex envelope of the electric field in the jth fibre, which is linearly coupled with its nearest neighbours. It is assumed that the number of fibres is finite. It must be emphasized that given the genericity of perturbed, non-integrable Hamiltonian systems, we believe that the general framework and techniques presented herein will be of value to very broad and diverse areas such as nonlinear optics, matter waves and plasma physics among others.

The detailed proof of the theoretical results presented herein will follow in a longer work [8]. However, we believe that it is useful and instructive to crystallize the findings and their essence, as well as their application in a communication of the present form. In the heart of these findings lies the necessity of looking at a reduced finite-dimensional eigenvalue problem which is relevant to finding the point spectrum eigenvalues created by the breaking of the symmetries of the original problem. In this type of centre manifold approach, through the eigenvalues and eigenvectors given in lemma 2.4, we can determine to leading order the effect of the perturbation to the eigenvalues and eigenvectors of the original system, as well as their Krein signature. Roughly speaking, the Krein signature of purely imaginary eigenvalues is a function of whether or not the underlying wave is a minimizer for the energy on the centre manifold in function space. If the wave is a minimizer, then the sign will always be positive; otherwise, eigenvalues with negative sign are allowed. The sign of an eigenvalue cannot change unless there is a collision with an eigenvalue of the opposite sign; hence, it yields global information about an eigenvalue. If eigenvalues of the opposite sign collide, then oscillatory instabilities generically ensue (the so-called Hamiltonian–Hopf bifurcation), whereas if eigenvalues of the same sign collide, then they will simply pass through each other [9].

The presentation of our work will be as follows: in the following section we will present the general framework and the main results, while in section 3 we will consider the specific case of the CNLS. Finally, section 4 summarizes our findings and presents our conclusions.

2. General theory

The results in this section relate the spectrum (denoted henceforth by σ) $\sigma(E''(\Phi))$ to $\sigma(JE''(\Phi))$, where Φ represents a solution to the steady-state problem $E'(u) = 0$. The operator $E''(\Phi)$ is self-adjoint; hence, $\sigma(E''(\Phi)) \subset \mathbb{R}$. Since $JE''(\Phi)$ is the composition of a skew-symmetric operator with a self-adjoint operator, one has the property that if $\lambda \in \sigma(JE''(\Phi))$, then $-\lambda, \pm\lambda^* \in \sigma(JE''(\Phi))$. Thus, the eigenvalues for $JE''(\Phi)$ come in quartets. Below one sees the manner in which the negative directions for $E''(\Phi)$ influence the unstable spectrum of $JE''(\Phi)$. It is most interesting to note that negative directions for $E''(\Phi)$ do *not* necessarily lead to an exponential instability of the wave. A detailed discussion of the proof of these results can be found in [8].

2.1. The unperturbed problem

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $\mathscr{G}$ be a finite-dimensional Abelian connected Lie group with Lie algebra $\mathfrak{g}$, and set $\dim(\mathfrak{g}) = n$. Denote by $e^{\omega} := \exp(\omega)$ the exponential map from $\mathfrak{g}$ to $\mathscr{G}$. Assume that T is a unitary representation of $\mathscr{G}$ on H, so that $T'(e)$ maps $\mathfrak{g}$ onto the space of closed skew-symmetric operators. Denote $T_\omega := T'(e)\omega$ as the generator of the semigroup $T(e^{\omega t})$, and note that T_ω is linear in $\omega \in \mathfrak{g}$. The group orbit $\mathscr{G}u$ is defined by $\mathscr{G}u := \{T(g)u : g \in \mathscr{G}\}$. Now assume that E is invariant under a group orbit, i.e., $E(T(g)u) = E(u)$ for all $g \in \mathscr{G}$ and $u \in H$. Define the functional $Q_\omega(u) := \frac{1}{2}\langle J^{-1}T_\omega u, u\rangle$, and note that $Q_\omega'' = J^{-1}T_\omega$ is a symmetric linear operator. Furthermore, Q_ω is invariant under a group orbit.

Consider the Hamiltonian system on H given by

$$\frac{dv}{dt} = JE'(v).$$

We are interested in relative equilibria of this system, i.e., solutions which satisfy $u(t) \in \mathscr{G}u(0)$ for all t. A relative equilibrium satisfies $u(t) = T(e^{\omega t})u(0)$ for some $\omega \in \mathfrak{g}$. Changing variables via

$$v(t) = T(\exp(\omega t))u(t)$$

yields the system

$$\frac{du}{dt} = JE_0'(u;\omega) \tag{2.1}$$

where

$$E_0'(u;\omega) := E'(u) - J^{-1}T_\omega u.$$

Relative equilibria will then be critical points of the functional $E_0(u;\omega) := E(u) - Q_\omega(u)$ for some $\omega \in \mathfrak{g}$.

We assume that the steady-state equation

$$E_0'(u;\omega) = 0$$

has a smooth family $\Phi(\omega)$ of solutions, where ω varies in $\mathfrak{g}$. Furthermore, we assume that the isotropy subgroups $\{g \in \mathscr{G} : T(g)\Phi(\omega) = \Phi(\omega)\}$ are discrete. This assumption implies that the group orbits $\mathscr{G}\Phi(\omega)$ have dimension n for each fixed $\omega \in \mathfrak{g}$. Since $\mathscr{G}$ is Abelian, for each fixed $\omega \in \mathfrak{g}$ the entire group orbit $T(g)\Phi(\omega)$ consists of relative equilibria with time evolution $T(e^{\omega t})$.

Let the linear operator about the wave Φ be denoted by JE_0''. Fix a basis $\{\omega_1, \ldots, \omega_n\}$ that satisfies the property that the set $\{T_{\omega_1}\Phi, \ldots, T_{\omega_n}\Phi\}$ is orthogonal. One has that $E_0''T_{\omega_j}\Phi = 0$ for $j = 1, \ldots, n$ [8, section 2]. Since $\mathscr{G}$ is Abelian, under the nondegeneracy condition that D_0 is nonsingular, where D_0 is defined in (2.3), it is known that the operator JE_0'' will have a nontrivial kernel:

$$JE_0''(\Phi)T_{\omega_i}\Phi = 0 \qquad JE_0''(\Phi)\partial_{\omega_i}\Phi = T_{\omega_i}\Phi \tag{2.2}$$

for $i = 1, \ldots, n$, with $\partial_\omega := \partial/\partial\omega$. Furthermore, this set is a basis for the kernel [8, section 3.1]. It is interesting to note that the solutions to the above linear system yield not only a basis for the tangent space to the group orbit, but also a basis for the tangent space of the manifold of relative equilibria. It will be assumed that

Assumption 2.1. *The linear operator* E_0'' *is Fredholm of index zero. If one sets*

$$Z = \mathrm{Span}\left\{T_{\omega_1}\Phi, \ldots, T_{\omega_n}\Phi\right\}$$

then $H = N \oplus Z \oplus P$*, where N is the finite-dimensional subspace*

$$N = \{u \in H : \langle u, E_0''u\rangle < 0\}$$

and $P \subset H$ *is a closed subspace such that*

$$\langle u, E_0''u\rangle > \delta\langle u, u\rangle \qquad u \in P$$

for some constant $\delta > 0$.

Set

$$H_1 := \{u \in H : \left\langle u, E_0''\partial_{\omega_i}\Phi\right\rangle = 0, i = 1, \ldots, n\}.$$

It is shown in [10] that when solving the linear eigenvalue problem $JE_0''(\Phi)u = \lambda u$, it is sufficient to consider only those $u \in H_1$. This also follows from a standard solvability theory, as $J^{-1}T_{\omega_i}\Phi = E_0''\partial_{\omega_i}\Phi$ are solutions to the adjoint eigenvalue problem at $\lambda = 0$ for $i = 1, \ldots, n$. Define the symmetric matrix $D_0 \in \mathbb{R}^{n\times n}$ by

$$(D_0)_{ij} = \left\langle \partial_{\omega_j}\Phi, E_0''\partial_{\omega_i}\Phi\right\rangle. \tag{2.3}$$

For a given self-adjoint operator A, let $\mathrm{n}(A)$ denote the number of negative eigenvalues, $\mathrm{p}(A)$ be the number of positive eigenvalues and $\mathrm{z}(A)$ the number of zero eigenvalues.

Lemma 2.2 [10]**.** *Suppose that* $\mathrm{z}(D_0) = 0$*. The operator* E_0'' *restricted to the space* H_1 *has the negative index*

$$\mathrm{n}(E_0''|_{H_1}) = \mathrm{n}(E_0'') - \mathrm{n}(D_0).$$

If $\mathrm{n}(E_0''|_{H_1}) = 0$*, then the wave is a local minimizer for the energy* $E_0(u)$*, and is therefore stable.*

The interpretation of lemma 2.2 is as follows. Suppose that the operator E_0'' satisfies $\mathrm{n}(E_0'') = k \geqslant 1$. One then has that the wave is not a local minimizer for E_0. However, there are conserved quantities associated with the evolution equation, and it is possible that these quantities may 'knock out' some or all of the unstable directions. In fact, there are exactly $\dim(\mathfrak{g})$ conserved quantities, and they are given by

$$Q_i(u) := \tfrac{1}{2}\left\langle J^{-1}T_{\omega_i}u, u\right\rangle \qquad i = 1, \ldots, n.$$

The quantity $\mathrm{n}(D_0)$ precisely determines the number of directions which are eliminated by the conserved quantities. Hence, $\mathrm{n}(E_0'') - \mathrm{n}(D_0)$ determines the number of unstable directions for the energy after the constraints have been taken into account.

2.2. The perturbed problem

In this and the subsequent sections it will be assumed that the energy is of the form $E_0(u) + \epsilon E_1(u)$, where $0 < \epsilon \ll 1$. It will be assumed that the perturbation breaks $1 \leqslant k_\mathrm{s} \leqslant n$ of the original symmetries, so that the perturbed system will have $n - k_\mathrm{s}$ symmetries. It will be further assumed that the problem is well understood for $\epsilon = 0$; in particular, the calculation in lemma 2.2 has been made. When considering the existence question, one has the following:

Lemma 2.3 [7, 8]**.** *A necessary condition for persistence of the wave is*

$$\left\langle E_1'(\Phi(\omega)), T_{\omega_j}\Phi\right\rangle = 0 \qquad j = 1, \ldots, n \tag{2.4}$$

for some $\omega \in \mathfrak{g}$. *The condition is sufficient if* $\mathrm{z}(M) = n - k_\mathrm{s}$, *where the symmetric matrix* M *satisfies*

$$M_{ij} := \left\langle T_{\omega_i}\Phi, E_1''(\Phi(\omega))T_{\omega_j}\Phi\right\rangle.$$

Since the perturbation breaks k_s symmetries, and the system is Hamiltonian, $2k_\mathrm{s}$ eigenvalues will leave the origin. The following lemma, which tracks these small eigenvalues, was proved in [8] via a Lyapunov–Schmidt reduction.

Lemma 2.4. *The* $\mathrm{O}(\sqrt{\epsilon})$ *eigenvalues and associated eigenfunctions for the perturbed problem are given by*

$$\lambda = \sqrt{\epsilon}\lambda_1 + \mathrm{O}(\epsilon) \qquad u = \sum_{i=1}^{n} v_i\left(T_{\omega_i}\Phi + \sqrt{\epsilon}\lambda_1\partial_{\omega_i}\Phi\right) + \mathrm{O}(\epsilon)$$

where λ_1 *is the eigenvalue and* $\mathbf{v}$ *is the associated eigenvector for the generalized eigenvalue problem*

$$\left(D_0\lambda_1^2 + M\right)\mathbf{v} = \mathbf{0}.$$

Remark 2.5. It should be noted that the above eigenvalue problem will have $2(n - k_\mathrm{s})$ zero eigenvalues, due to the fact that $n - k_\mathrm{s}$ symmetries are assumed to be preserved under the perturbation.

If an eigenvalue has nonzero real part, the Krein signature is zero [6, 11]. The Krein signature of a purely imaginary $\mathrm{O}(\sqrt{\epsilon})$ eigenvalue given in lemma 2.4 is

$$K = \mathrm{sign}(\mathbf{v}^\mathrm{T} M\mathbf{v}) = \mathrm{sign}(\mathbf{v}^\mathrm{T} D_0\mathbf{v}) \tag{2.5}$$

where $\mathbf{v}$ is the associated eigenvector [8]. It may also be possible for eigenvalues to pop out of the essential spectrum, creating internal modes via an *edge bifurcation* [12, 13]. Since these eigenvalues will be of O(1), they will not be captured by the perturbation expansion given in lemma 2.4 . However, as is seen in theorem 2.6, this is not problematic. There it is seen that any O(1) eigenvalues will be purely imaginary with positive Krein sign, and hence for small ϵ do not contribute to any instability.

In the statement of the below theorem the symmetric matrix D_ϵ is defined by

$$(D_\epsilon)_{ij} := \mathbf{w}_i^\mathrm{T} D_0\mathbf{w}_j \tag{2.6}$$

where the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-k_\mathrm{s}}\}$ is a basis for $\ker(M)$. The following is proved in [8] regarding $\sigma(J(E_0'' + \epsilon E_1''))$ for $0 < \epsilon \ll 1$.

Theorem 2.6. *Suppose that the unperturbed wave is stable, i.e.,* $\mathrm{n}(E_0'') = \mathrm{n}(D_0)$. *Let* k_r *represent the number of real positive eigenvalues,* $2k_\mathrm{c}$ *the number of complex eigenvalues with positive real part and* $2k_\mathrm{i}$ *the number of purely imaginary eigenvalues with negative Krein signature for the perturbed problem (counting multiplicity). Assume that* $\mathrm{z}(D_\epsilon) = 0$. *Then*

$$k_\mathrm{r} + 2k_\mathrm{c} + 2k_\mathrm{i} = \mathrm{n}(E_0'') + \mathrm{n}(M) - \mathrm{n}(D_\epsilon).$$

Furthermore, all of these eigenvalues are of $\mathrm{O}(\sqrt{\epsilon})$, *and*

$$k_\mathrm{s} \geqslant k_\mathrm{r} \geqslant |\mathrm{n}(M) - (\mathrm{n}(D_0) - \mathrm{n}(D_\epsilon))|.$$

Any eigenvalues arising from an edge bifurcation will be purely imaginary with positive Krein signature.

Remark 2.7. One has that

(a) the upper bound on k_r arises from the facts that there are only $2k_s$ eigenvalues of $O(\sqrt{\epsilon})$ and the system is Hamiltonian;
(b) since $n(D_\epsilon) \leqslant n(D_0) = n(E_0'')$, the perturbed wave cannot be a minimizer unless $n(M) = 0$ and $n(D_\epsilon) = n(D_0)$.

One possible interpretation of theorem 2.6 is as follows. As previously mentioned, for the unperturbed problem each unstable direction associated with E_0'' is neutralized by an invariance, which in turn are each generated by a symmetry. Now, D_ϵ is the representation of D_0 when restricted to the symmetry group which persists upon the perturbation. The quantity

$$n(E_0'') - n(D_\epsilon) = n(D_0) - n(D_\epsilon)$$

then precisely details the number of unstable directions associated with E_0'' which are no longer neutralized by the invariances. The quantity $n(M)$ is the number of additional unstable directions generated by the symmetry-breaking perturbation E_1. The theorem essentially illustrates that the number of potentially unstable eigenvalue pairs in the system is obtained by keeping track of these eigendirections.

3. Case example: the CNLS

We will now apply the theoretical results to a particular problem. For $j = 1, \ldots, N$ consider the system

$$i\partial_t q_j + \frac{1}{2}\big(\partial_x^2 - 1\big)q_j + |q_j|^2 q_j = -\frac{1}{2}\epsilon \sum_{i=1}^{N} \kappa_{ij} q_i \qquad (3.1)$$

where $\kappa = (\kappa_{ij}) \in \mathbb{R}^{N\times N}$ is symmetric. Furthermore, the coupling constants satisfy $\kappa_{ij} \geqslant 0$. The set of equations (3.1) has been extensively used as a model of linearly coupled optical fibres [14–18], where the small parameter $0 < \epsilon \ll 1$ represents the strength of the coupling between the fibres. The system is Hamiltonian, where the Hamiltonian is given by $E_0 + \epsilon E_1$ with

$$E_0(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{N}\int_{-\infty}^{+\infty} (|q_{i,x}|^2 + |q_i|^2 - |q_i|^4)\,dx$$

and

$$E_1(\mathbf{q}) = -\frac{1}{2}\sum_{i,j=1}^{N}\int_{-\infty}^{+\infty} \kappa_{ij}(q_i q_j^* + q_i^* q_j)\,dx.$$

This system has been extensively studied both numerically and analytically in the case that $\kappa_{j\pm1,j} = 1$ for all j and $\kappa_{ij} = 0$ otherwise. The early works of Aceves *et al* [14] and Weinstein *et al* [15] demonstrated that for sufficiently small coupling ϵ the ground state of the system is the one in which all the 'power' resides in one fibre (the so-called collapse-effect compressor). A number of other works considered details of the problem such as the existence of solutions [16] or the multiple pulse interaction dynamics through a variational method [17, 18].

Even though the global minimizer of the system's Hamiltonian is known from these earlier works, we will examine here how the instability of various configurations occurs and will give specific predictions about the growth rate of the corresponding instabilities. These predictions are important in appreciating the different instabilities arising for different solutions, as well as in assessing how rapidly these instabilities set in and destroy the configurations of interest.

Set $\Phi(x) = \text{sech}(x)$. When $\epsilon = 0$ one has the solution

$$q_j(x) = \delta_j \Phi(x + x_j)\,\mathrm{e}^{\mathrm{i}\theta_j} \qquad \delta_j \in \{0, 1\}. \tag{3.2}$$

To simplify the discussion, it will be assumed that $\delta_j = 1$ for all j. Set $\mathbf{Q}_0 = (\Phi, \ldots, \Phi)^{\mathrm{T}} \in \mathbb{R}^N$. When $\epsilon = 0$ solutions to equation (3.1), and in particular the wave $\mathbf{Q}_0$, are invariant under the action

$$T(x_1, \ldots, x_N; \theta_1, \ldots, \theta_N)\mathbf{Q}_0 = (\Phi(x + x_1)\,\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \Phi(x + x_N)\,\mathrm{e}^{\mathrm{i}\theta_N})^{\mathrm{T}}.$$

The parameters x_j and θ_j play the role of the elements ω_j mentioned in the previous section. Note that for $\epsilon > 0$ the perturbed wave $\mathbf{Q}_\epsilon$ will be invariant only under the action

$$T(x_0, \theta_0)\mathbf{Q}_\epsilon = \mathbf{Q}_\epsilon(x + x_0)\,\mathrm{e}^{i\theta_0}. \tag{3.3}$$

Hence, the perturbation breaks $2N - 2$ symmetries associated with the original system. In order to determine the stability of the perturbed wave, one must then minimally track $2N - 2$ pairs of $\mathrm{O}(\sqrt{\epsilon})$ eigenvalues for $\epsilon > 0$ small.

3.1. Existence of waves

It was seen in lemma 2.3 that the existence of waves for the perturbed system can be determined by looking for critical points of the reduced Hamiltonian

$$H(\mathbf{x}, \Theta) := E_1(T(\mathbf{x}; \Theta)\mathbf{Q}_0)$$

where $\mathbf{x} = (x_1, \ldots, x_N)$, $\Theta = (\theta_1, \ldots, \theta_N)$. For the system (3.1) the reduced Hamiltonian is given by

$$H(\mathbf{x}, \Theta) = -\sum_{i,j=1}^{N} \kappa_{ij} \cos(\Delta\theta_{ij}) F(\Delta x_{ij}) \tag{3.4}$$

where

$$\Delta\theta_{ij} = \theta_i - \theta_j \qquad \Delta x_{ij} = x_i - x_j$$

and

$$F(y) := \int_{-\infty}^{+\infty} \Phi(x + y)\Phi(x)\,\mathrm{d}x = \frac{2y}{\sinh(y)}. \tag{3.5}$$

For persistence of the wave, one must study the set of equations given in equation (2.4), which is a system of $2N$ equations in the variables $(\mathbf{x}, \Theta)$. Upon using the definition of the derivative and expanding via a Taylor polynomial one immediately sees that these equations are equivalent to the set

$$\partial_{x_i} H(\mathbf{x}, \Theta) = \partial_{\theta_i} H(\mathbf{x}, \Theta) = 0 \tag{3.6}$$

for $i = 1, \ldots, N$. The existence equations can then be written for $i = 1, \ldots, N$ as

$$\begin{aligned} \partial_{x_i} : &\quad -\sum_{j \neq i} \kappa_{ij} \cos(\Delta\theta_{ij}) F'(\Delta x_{ij}) = 0 \\ \partial_{\theta_i} : &\quad \sum_{j \neq i} \kappa_{ij} \sin(\Delta\theta_{ij}) F(\Delta x_{ij}) = 0. \end{aligned}$$

The following result is an easy consequence of the existence equations.

Lemma 3.1. *One set of solutions to equation (3.6) is*

$$\Delta x_{ij} = 0 \qquad \Delta\theta_{ij} \in \{0, \pi\}$$

for all i, j.

Now assume that $\kappa_{ij} = 0$ for $|i - j| \geqslant 2 \pmod N$, i.e., the coupling between the fibres is nearest neighbour only. Note that this assumption implicitly assumes that the array is circular, i.e., fibre N is linearly coupled to fibre 1. Under this assumption, equation (3.6), upon keeping in mind the fact that the sum is taken over j with i being fixed, and upon using the facts that $F'(\cdot)$ and $\sin(\cdot)$ are odd, can be written as

$$\begin{aligned} \partial_{x_i}: &\quad \kappa_{i,i-1}\cos(\Delta\theta_{i,i-1})F'(\Delta x_{i,i-1}) = \kappa_{i+1,i}\cos(\Delta\theta_{i+1,i})F'(\Delta x_{i+1,i}) \\ \partial_{\theta_i}: &\quad \kappa_{i,i-1}\sin(\Delta\theta_{i,i-1})F(\Delta x_{i,i-1}) = \kappa_{i+1,i}\sin(\Delta\theta_{i+1,i})F(\Delta x_{i+1,i}). \end{aligned} \tag{3.7}$$

Note that the solutions to equation (3.7) must satisfy the constraints

$$\sum_{i=1}^{N} \Delta x_{i,i-1} = 0 \qquad \sum_{i=1}^{N} \Delta\theta_{i,i-1} = 0 \pmod{2\pi}. \tag{3.8}$$

First consider the solution set to equation (3.7) under the restriction that $\Delta\theta_{i+1,i} \in \{0, \pi\}$ for all i. The second set of equations in equation (3.7) is then automatically satisfied, and the first set can be rewritten as

$$\kappa_{i,i-1}\cos(\Delta\theta_{i,i-1})F'(\Delta x_{i,i-1}) = \kappa_{1,N}\cos(\Delta\theta_{1,N})F'(\Delta x_{1,N})$$

for $i = 2, \ldots, N$. Upon using the fact that $F'(\cdot)$ is odd, it is clear that one set of solutions is $\Delta x_{i,i-1} = 0$ for all i. Other interesting solutions may also be possible. For example, suppose that $\kappa_{i,i-1} = 1$ for all i, and that $\Delta x_{1,N} > 0$. If one assumes that $\Delta\theta_{1,N} = \pi$, then the above equation is solved if $|\Delta x_{i,i-1}| = \Delta x_{1,N}$ with $\mathrm{sign}(\Delta x_{i,i-1}) = -\mathrm{sign}(\cos(\Delta\theta_{i,i-1}))$. Upon setting $\delta_i = -\mathrm{sign}(\Delta x_{i,i-1})/\,\mathrm{sign}(\cos(\Delta\theta_{i,i-1}))$, so that $\Delta x_{i,i-1} = \delta_i \Delta x_{1,N}$, one sees that the first constraint in equation (3.8) reads

$$\Delta x_{1,N} \sum_{i=1}^{N} \delta_i = 0.$$

This equation may certainly be satisfied if N is even. We will not pursue this avenue any further in this paper, and will leave this as a topic for a future study.

Alternatively, consider the solution set to equation (3.7) under the restriction that $\Delta x_{i,i-1} = 0$. The first set of equations then automatically holds, and the second set can be rewritten as

$$\kappa_{i,i-1}\sin(\Delta\theta_{i,i-1}) = \kappa_{1,N}\sin(\Delta\theta_{1,N}) \qquad i = 2, \ldots, N.$$

It is clear that one set of solutions is $\Delta\theta_{i,i-1} \in \{0, \pi\}$. If $\kappa_{i,i-1} = 1$ for all i, then upon setting $\Delta\theta_{1,N} = \pi(1 + \alpha)/2$ for some $\alpha \in [-1, 1]$ one sees that for each $i = 1, \ldots, N$, $\Delta\theta_{i,i-1} = \pi(1 + \delta_i\alpha)/2$, where $\delta_i \in \{-1, +1\}$. The constraint in equation (3.8) then reads

$$\sum_{i=1}^{N} \Delta\theta_{i,i-1} = \frac{\pi}{2}\left(N + \alpha\sum_{i=1}^{N}\delta_i\right) \pmod{2\pi}.$$

Upon setting $M := \sum \delta_i \in \mathbb{Z}$, one has that equation (3.8) will be satisfied if and only if for some $J \in \mathbb{Z}$,

$$\alpha = \frac{4J - N}{M} \in \mathbb{Q}.$$

Note that the condition $|\alpha| \leqslant 1$ yields $N - |M| \leqslant 4J \leqslant N + |M|$. It is straightforward to check that one needs $N \geqslant 3$ in order to achieve $|\alpha| \neq 1$. For example, if $\delta_i = 1$ for each i, then one sees that permissible values are

$$\alpha \in \left\{ -1, \frac{4-N}{N}, \frac{8-N}{N}, \ldots, 1 \right\}$$

i.e.,

$$\Delta\theta_{i,i-1} = 2\pi \frac{J}{N} \qquad J = 0, \ldots, \frac{N}{2}.$$

If the constraint $\kappa_{i,i-1} = 1$ for all i is lifted, then other interesting solutions may be possible. These topics will also be left for a future study.

Finally, if one chooses $N = 4L$ for some $L \in \mathbb{N}$ with $\Delta\theta_{i,i-1} = \pi/2$, then it is easy to see that equation (3.6) is simplified, and that the second part of equation (3.8) is automatically satisfied. The restriction given below on the interaction matrix κ can be relaxed to $\kappa_{i,i-1} = \kappa_0$ for all i; however, we can choose $\kappa_0 = 1$ without loss of generality.

Lemma 3.2. *Assume that $N = 4L$ for $L \in \mathbb{N}$, and further assume that $\kappa_{i,i-1} = 1$ for all i. Regarding the solutions to equation (3.6), if*

$$\Delta\theta_{i+1,i} = \frac{\pi}{2} \qquad i = 1, \ldots, N$$

then

$$\Delta x_{i+1,i} = \pm\Delta x_{i,i-1} \qquad i = 1, \ldots, N.$$

Proof. Under the assumptions on $\Delta\theta_{i+1,i}$ and $\kappa_{i,i-1}$, equation (3.7) reduces to

$$F(\Delta x_{i,i-1}) = F(\Delta x_{i+1,i}).$$

Since $F(\cdot)$ is positive, monotonically decreasing for positive arguments and even, one immediately gets the conclusion regarding the translation difference that $\Delta x_{i,i-1} = \pm\Delta x_{i+1,i}$. □

Remark 3.3. One implicitly assumes in the above that the first part of equation (3.8) is satisfied in choosing $\Delta x_{i+1,i}$.

The physical interpretation of the statement is that each pair of adjacent pulses can be separated by the same or opposite distance between the pulse centres as the previous/next pair. Hence the pulse centres can form any number of nodes of a lattice of 'separations'. The physical intuition behind this result is that if the two adjacent sets of pairs have equal distances between the pulse centres, then there is no net flux of particles, and the configuration is stationary because the flux from the left equals the flux to the right. In contrast, if $\Delta x_{i,i-1} = -\Delta x_{i+1,i}$ then the fluxes are opposite and the net flux is zero, hence producing no current.

3.2. Stability of waves

When $\epsilon = 0$ the linearization is given by $\mathbf{JL}_0$, where $\mathbf{L}_0 = E_0''(\mathbf{Q}_0)$. Here

$$\mathbf{J} = \mathrm{diag}(J, \ldots, J) \in \mathbb{R}^{2N\times 2N} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\mathbf{L}_0 = \mathrm{diag}(L_0, \ldots, L_0) \in \mathbb{R}^{2N\times 2N} \qquad L_0 = \mathrm{diag}(L_\mathrm{r}, L_\mathrm{i})$$

with

$$L_{\mathrm{r}} = -\tfrac{1}{2}\big(\partial_x^2 - 1\big) - 3\Phi(x)^2 \qquad L_{\mathrm{i}} = -\tfrac{1}{2}\big(\partial_x^2 - 1\big) - \Phi(x)^2.$$

Since

$$L_{\mathrm{r}}\Phi_x = 0 \qquad L_{\mathrm{i}}\Phi = 0$$

and

$$L_{\mathrm{r}}(\Phi + x\Phi_x) = -2\Phi \qquad L_{\mathrm{i}}(x\Phi) = -\Phi_x$$

one has that upon writing the wave in real and imaginary parts that for each $j = 1, \ldots, N$,

$$T_{x_j}\mathbf{Q}_0 = \Phi_x \mathbf{e}_{2j-1} \qquad \partial_{x_j}\mathbf{Q}_0 = -x\Phi\mathbf{e}_{2j}$$

and

$$T_{\theta_j}\mathbf{Q}_0 = \Phi\mathbf{e}_{2j} \qquad \partial_{\theta_j}\mathbf{Q}_0 = \tfrac{1}{2}(\Phi + x\Phi_x)\mathbf{e}_{2j-1}.$$

Here $\mathbf{e}_m$ represents the mth unit vector. After a reordering of the basis for $\mathbb{R}^{2N}$ the matrix D_0 given in equation (2.3) is given by

$$\begin{aligned} D_0 &= \mathrm{diag}(\langle\Phi_x, -x\Phi\rangle I_N, -\langle\Phi, \Phi + x\Phi_x\rangle/2I_N) \\ &= \mathrm{diag}(I_N, -I_N). \end{aligned}$$

With this reordering of the basis the matrix M given in lemma 2.4 now satisfies

$$M_{ij} = \begin{cases} \partial^2_{x_i x_j} E_1 & 1 \leqslant i \leqslant N \quad 1 \leqslant j \leqslant N \\ \partial^2_{x_i \theta_j} E_1 & 1 \leqslant i \leqslant N \quad N+1 \leqslant j \leqslant 2N \\ \partial^2_{x_i \theta_j} E_1 & N+1 \leqslant i \leqslant 2N \quad 1 \leqslant j \leqslant N \\ \partial^2_{\theta_i \theta_j} E_1 & N+1 \leqslant i \leqslant 2N \quad N+1 \leqslant j \leqslant 2N. \end{cases} \tag{3.9}$$

Before continuing, let us apply the result of theorem 2.6. It is clear that $\mathrm{n}(D_0) = N$. Furthermore, since $\mathrm{n}(L_r) = 1$ and $\mathrm{n}(L_i) = 0$, one has that $\mathrm{n}(E_0'') = N$, so by lemma 2.2 the unperturbed wave is stable. As an application of theorem 2.6 we then have

Lemma 3.4. *Suppose that* $\mathrm{z}(D_\epsilon) = 0$. *For* $0 < \epsilon \ll 1$ *the number of unstable eigenvalues is*

$$k_{\mathrm{r}} + 2k_{\mathrm{c}} + 2k_{\mathrm{i}} = N - \mathrm{n}(D_\epsilon) + \mathrm{n}(M)$$

where k_{r} *is the number of positive real eigenvalues,* $2k_{\mathrm{c}}$ *is the number of complex eigenvalues with positive real part and* $2k_{\mathrm{i}}$ *is the number of purely imaginary eigenvalues with negative Krein signature. All of these eigenvalues are of* $\mathrm{O}(\sqrt{\epsilon})$. *Furthermore,*

$$N - 1 \geqslant k_{\mathrm{r}} \geqslant |\mathrm{n}(M) - (N - \mathrm{n}(D_\epsilon))|.$$

Remark 3.5. As a consequence of theorem 2.6, any eigenvalue arising from an edge bifurcation must be purely imaginary with positive Krein signature. Furthermore, although it will not be proved herein, it can be shown that at most N eigenvalues will arise from an edge bifurcation.

3.2.1. General interaction matrix. First, suppose that the perturbed wave is given by lemma 3.1; in particular, this implies that $\Delta x_{i,j} = 0$ for all i, j. Upon using the form of E_1 given in equation (3.4), and applying it to equation (3.9), it is seen that

$$M = \mathrm{diag}(-F''(0)A, F(0)A) \in \mathbb{R}^{2N\times 2N} \tag{3.10}$$

where the symmetric matrix $A \in \mathbb{R}^{N\times N}$ for $N \geqslant 3$ satisfies

$$A_{ij} := \begin{cases} -\kappa_{ij}\cos(\Delta\theta_{ij}) & i \neq j \\ \sum\limits_{k\neq i} \kappa_{ik}\cos(\Delta\theta_{ik}) & i = j. \end{cases} \tag{3.11}$$

If $N = 2$ the matrix A is given by

$$A = \kappa_{12}\cos(\Delta\theta_{12})\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Note that $\mathrm{n}(M) = 2\mathrm{n}(A)$. It is clear that $\gamma = 0$ is an eigenvalue of A with associated eigenvector $\mathbf{v} = (1, 1, \ldots, 1)^{\mathrm{T}}$. As such, since M is block diagonal one sees from the conclusion of lemma 2.4 that under the assumption $\mathrm{z}(A) = 1, \lambda = 0$ is an eigenvalue with geometric multiplicity 2 and algebraic multiplicity 4. This is simply a reflection of the fact that equation (3.1) is a Hamiltonian system with two symmetries.

Lemma 3.6. *Suppose that $\Delta x_{ij} = 0$ for all i, j and that $\mathrm{z}(A) = 1$. Letting $\gamma_i, i = 1, \ldots, N$, represent the eigenvalues of A, the $\mathrm{O}(\sqrt{\epsilon})$ eigenvalues of lemma 3.4 are given to leading order by*

$$\lambda_i^2 = \epsilon \begin{cases} -2\gamma_i/3 & i = 1, \ldots, N \\ 2\gamma_{i-N} & i = N+1, \ldots, 2N. \end{cases}$$

Furthermore, the statement of lemma 3.4 can be refined to read

$$k_{\mathrm{r}} = N - 1 \qquad k_{\mathrm{i}} = \mathrm{n}(A).$$

Proof. Under the assumption that $\mathrm{z}(A) = 1$, for $\epsilon > 0$ the remaining invariances are given in equation (3.3); thus, one has that a basis for $\ker(M)$ is given by $\{\mathbf{w}_1, \mathbf{w}_2\}$, where

$$\mathbf{w}_1 = \left(\sum_{i=1}^{N} \mathbf{e}_i, \mathbf{0}\right)^{\mathrm{T}} \qquad \mathbf{w}_2 = \left(\mathbf{0}, \sum_{i=1}^{N} \mathbf{e}_{N+i}\right)^{\mathrm{T}}.$$

As a consequence,

$$D_\epsilon = N\,\mathrm{diag}(1, -1)$$

so that $\mathrm{n}(D_\epsilon) = 1$ and $\mathrm{z}(D_\epsilon) = 0$. Hence, the statement of lemma 3.4 can be refined to read

$$k_{\mathrm{r}} + 2k_{\mathrm{c}} + 2k_{\mathrm{i}} = N - 1 + 2\mathrm{n}(A) \tag{3.12}$$

and

$$N - 1 \geqslant k_{\mathrm{r}} \geqslant |(N-1) - 2\mathrm{n}(A)|.$$

By lemma 2.4 the eigenvalue equation is

$$\left(D_0\lambda_1^2 + M\right)\mathbf{v} = \mathbf{0}.$$

Upon setting $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)^{\mathrm{T}}$, and using the facts that $F(0) = 2$, $F''(0) = -2/3$, the above equation can be rewritten as

$$\left(3\lambda_1^2 I_N + 2A\right)\mathbf{v}_1 = \mathbf{0} \qquad \left(-\lambda_1^2 I_N + 2A\right)\mathbf{v}_2 = \mathbf{0}. \tag{3.13}$$

The equation with $\mathbf{v}_1$ is the bifurcation equation for the translation eigenvalues, while that for $\mathbf{v}_2$ is that for the rotation eigenvalues. If $\gamma_i, i = 1, \ldots, N$, represent the eigenvalues of A, then the eigenvalues for equation (3.13) are given by

$$\lambda_{1,i}^2 = \begin{cases} -2\gamma_i/3 & i = 1, \ldots, N \\ 2\gamma_{i-N} & i = N+1, \ldots, 2N. \end{cases} \tag{3.14}$$

The structure of the matrices D_0 and M guarantees that each nonzero eigenvalue for A will result in both a real pair and an imaginary pair of eigenvalues for the linearized system. To leading order in the perturbation expansion there will be exactly $N - 1$ pairs of real eigenvalues, and $N - 1$ pairs of purely imaginary eigenvalues, arising from the symmetry-breaking perturbation. Now consider the Krein signature of the bifurcating eigenvalues. From equation (2.5) and the form of D_0 one has that for each imaginary eigenvalue given in equation (3.14),

$$K = \text{sign}(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2).$$

Thus, it is clear that for $i = 1, \ldots, N$ purely imaginary eigenvalues will have positive Krein sign, while for $i = N + 1, \ldots, 2N$ these eigenvalues will have negative sign.

The above calculation demonstrates that to leading order,

$$k_{\rm i} = \text{n}(A) \qquad k_{\rm r} = N - 1 \qquad k_{\rm c} = 0.$$

However, it is possible that to higher order the purely imaginary eigenvalue has a real part of $\text{O}(\epsilon)$, and the purely real eigenvalue has an imaginary part of $\text{O}(\epsilon)$. As is seen in [8, section 6.3], because the perturbation E_1 is reversible, one can rule out these scenarios. □

Remark 3.7. From a physical viewpoint, upon observation of equation (3.14) it is seen that for purely imaginary eigenvalues, those created from the breaking of the translation symmetry will have positive sign, while those arising from the breaking of the rotation symmetry will have negative sign.

3.2.2. Nearest-neighbour interaction. Now consider the special case that $\kappa_{ij} = 0$ for $|i - j| \geqslant 2 \pmod N$. As before, it is being implicitly assumed that the array is circular. The matrix M will have a form similar to that given in equation (3.10); however, the matrix A will have more structure. Setting

$$a_i := \kappa_{i,i-1} \cos(\Delta\theta_{i,i-1}) \tag{3.15}$$

the symmetric matrix $A \in \mathbb{R}^{N\times N}$ will be given by

$$A_{ij} := \begin{cases} -a_{i+1} & j = i + 1 \pmod N \\ a_i + a_{i+1} & j = i \\ 0 & \text{otherwise} \end{cases}$$

where the notation $a_{N+1} := a_1$ is being used.

The matrix A constructed in the above fashion is almost tri-diagonal. In the appendix the spectrum of such matrices is discussed, and in particular a relationship between the number of negative eigenvalues and the phase change between adjacent pulses is determined. The following result is direct consequence of lemma 3.6.

Lemma 3.8. *Consider the solutions given in lemma 3.1 with* $\kappa_{ij} = 0$ *for* $|i - j| \geqslant 2 \pmod N$. *For the variables defined in equation (3.15) suppose that the nondegeneracy condition*

$$\sum_{i=1}^{N}\left(\prod_{j\neq i} a_j\right) \neq 0$$

is satisfied. One then has that $\text{z}(A) = 1$, *so that the conclusion of lemma 3.6 holds. Furthermore, if* $\Delta\theta_{i,i-1} = 0$ *for all i, then A is positive definite, and if* $\Delta\theta_{i,i-1} = \pi$ *for all i, then A is negative definite. Otherwise,* $\text{n}(A) \geqslant 1$, *and if* $\Delta\theta_{1,N} = \pi$, *then*

$$\#(i \geqslant 2 : \Delta\theta_{i,i-1} = \pi) \leqslant \text{n}(A) \leqslant 1 + \#(i \geqslant 2 : \Delta\theta_{i,i-1} = \pi)$$

while if $\Delta\theta_{1,N} = 0$, *then*

$$\#(i \geqslant 2 : \Delta\theta_{i,i-1} = \pi) - 1 \leqslant \text{n}(A) \leqslant \#(i \geqslant 2 : \Delta\theta_{i,i-1} = \pi).$$

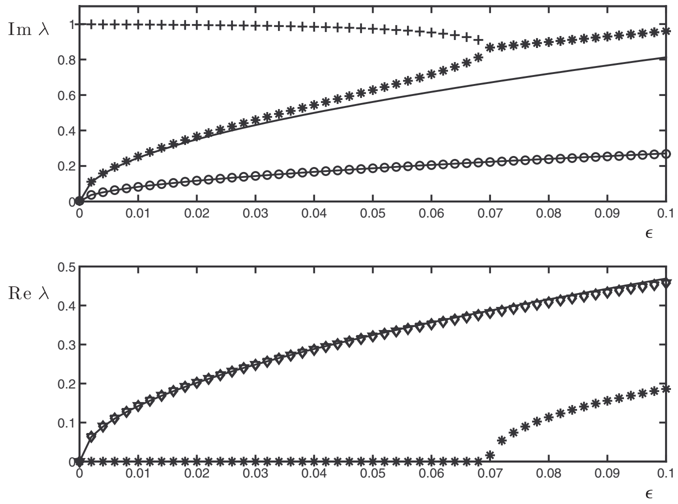

Figure 1. The top panel shows $\mathrm{Im}\,\lambda$ as a function of ϵ, while the bottom panel shows $\mathrm{Re}\,\lambda$ for the case of $(\theta_1, \theta_2, \theta_3) = (0, \pi, 0)$. The circles in the top panel show the smaller bifurcating imaginary eigenvalue of positive Krein sign, while the stars indicate the larger bifurcating imaginary eigenvalue of negative Krein sign. When $\epsilon \approx 0.07$ an eigenvalue with positive sign arising from an edge bifurcation (denoted by plus symbols) collides with the eigenvalue of negative sign, leading to the creation of a complex quartet. The bottom panel shows by triangles and diamonds the two pairs of bifurcating real eigenvalues and by stars, the real part of the negative Krein signature eigenvalue. Solid lines show the theoretical predictions of equation (3.14) for the different eigenvalues. Very good agreement is observed between the theoretical predictions and the full numerical results for small ϵ.

Remark 3.9. If

$$\sum_{i=1}^{N}\left(\prod_{j\neq i} a_j\right) = 0$$

then $\mathrm{z}(A) \geqslant 2$, so that there is either a hidden symmetry which is nongeneric, or there are some eigenvalues which will be of $\mathrm{O}(\epsilon)$ and which will not be captured by the perturbation expansion.

As an illustration of the theory to this point, consider figure 1. We set $\kappa_{j\pm1,j} = 1$, $\kappa_{jj} = -2$, and $\kappa_{ij} = 0$ otherwise, and using the notation of equation (3.2) we started with the pulse $(\theta_1, \theta_2, \theta_3) = (0, \pi, 0)$. The matrix A has the simple eigenvalues $\gamma = -3, 0, 1$. Lemma 3.8 implies that for small ϵ there will be a pair of imaginary eigenvalues with negative Krein sign, and another small pair with positive sign. It is seen that for $\epsilon \approx 0.07$ the imaginary eigenvalue with negative sign collides with an imaginary eigenvalue of positive sign which arises from an edge bifurcation. The collision leads to the creation of a complex quartet of eigenvalues.

In figure 2 we started with the pulse $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$. The matrix A has the eigenvalues $\gamma = 0, 3, 3$, which by lemma 3.8 implies that for small ϵ both pairs of imaginary eigenvalues will have positive Krein sign. Note also that via lemma 3.6 the unstable eigenvalues are purely real, which agrees with the numerics. Once again, very good agreement is obtained between the bifurcations predicted analytically from equation (3.14) and those observed numerically in figure 2.

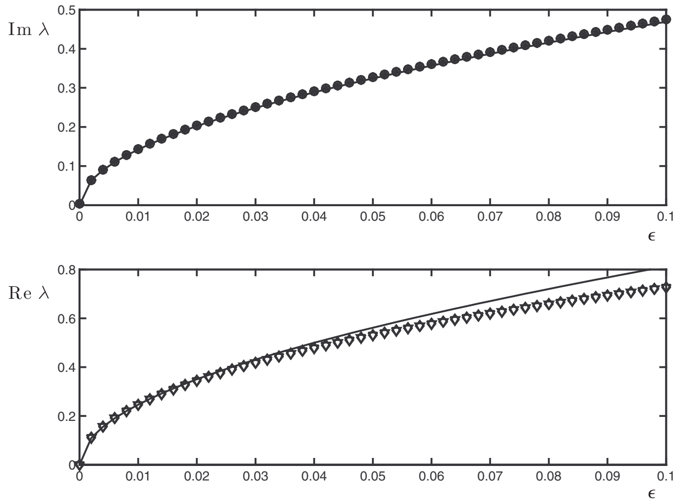

Figure 2. Same as figure 1, but for the case of $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$. The stars and circles in the top panel indicate the two pairs of eigenvalues bifurcating along the imaginary axis; the triangles and diamonds in the bottom panel indicate the ones bifurcating along the real axis. The solid lines denote the theoretical predictions which are again very successful in capturing the numerical results.

3.2.3. $\Delta\theta_{i+1,i} = \pi/2$. Let us now consider the perturbed wave given in lemma 3.2. In this case the stability matrix M is given by

$$M = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}.$$

Setting

$$b_i := F'(\Delta x_{i-1,i})$$

the symmetric matrix $B \in \mathbb{R}^{4L\times 4L}$ is given by

$$B_{ij} := \begin{cases} -b_{i+1} & j = i+1 \pmod{4L} \\ b_i + b_{i+1} & j = i \\ 0 & \text{otherwise} \end{cases}$$

where the notation $b_{4L+1} := b_1$ is being used. The appendix applies to the matrix B. The statement of lemma 3.2 can be restated to say that $\Delta x_{i,i-1} = \pm\Delta x_0$ for all i, where Δx_0 is a fixed translation. First note that if $\Delta x_0 = 0$, then B is the zero matrix; henceforth, it will be assumed that $\Delta x_0 \neq 0$. Equation (3.8), along with the fact that $F'(\cdot)$ is odd, implies that

$$\sum_{i=1}^{N} \left(\prod_{j\neq i} b_j \right) = 0$$

so that $\mathrm{z}(B) \geqslant 2$.

As in the proof of lemma 3.6, the structure of M guarantees that $\mathrm{n}(D_\epsilon) = \mathrm{z}(B)$. Furthermore, the structure of M requires that

$$\mathrm{n}(M) = 4L - \mathrm{z}(B).$$

Thus, the conclusion of lemma 3.4 reads

$$k_\mathrm{r} + 2k_\mathrm{c} + 2k_\mathrm{i} = 8L - 2\mathrm{z}(B).$$

With $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)^{\mathrm{T}}$ the eigenvalue equation $\left(D_0\lambda_1^2 + M\right)\mathbf{v} = \mathbf{0}$ can be rewritten as

$$\lambda_1^2\mathbf{v}_1 + B\mathbf{v}_2 = \mathbf{0} \qquad B\mathbf{v}_1 - \lambda_1^2\mathbf{v}_2 = \mathbf{0}$$

which in turn can be rewritten as

$$\left(B^2 + \lambda_1^4 I\right)\mathbf{v}_1 = \mathbf{0}.$$

Since B is symmetric, and hence diagonalizable, if γ_j for $j = 1, \ldots, 4L$ represents the real eigenvalues of B, then the eigenvalues λ satisfy

$$\lambda_{1,j}^2 = \pm\mathrm{i}|\gamma_j|. \tag{3.16}$$

Thus, all the eigenvalues are complex with nonzero real part. By this calculation one can immediately conclude that

$$k_{\mathrm{r}} = k_{\mathrm{i}} = 0 \qquad k_{\mathrm{c}} = 4L - \mathrm{z}(B).$$

Lemma 3.10. *Consider the wave given in lemma 3.2. Suppose that $\Delta x_{i,i-1} = \pm\Delta x_0$ with $\Delta x_0 \neq 0$. The $\mathrm{O}(\sqrt{\epsilon})$ eigenvalues satisfy*

$$k_{\mathrm{r}} = k_{\mathrm{i}} = 0 \qquad k_{\mathrm{c}} = 4L - \mathrm{z}(B).$$

Up to leading order the location of the nonzero eigenvalues is given by equation (3.16).

Remark 3.11. It may be the case that $\mathrm{z}(B) = 2$ is due to an extra internal symmetry. For example, suppose that $\Delta x_{2i,2i-1} = \Delta x_0$, while $\Delta x_{2i+1,2i} = -\Delta x_0$. In this case a solution for all $\epsilon \geqslant 0$ is given by $T(x_1, x_2; \theta_1, \theta_2)\mathbf{Q}_\epsilon$, which satisfies the property

$$q_{2k+1}(\cdot + x_1)\,\mathrm{e}^{\mathrm{i}\theta_1} = -q_{2k-1}(\cdot + x_1)\,\mathrm{e}^{\mathrm{i}\theta_1}$$

and

$$q_{2k}(\cdot + x_2)\,\mathrm{e}^{\mathrm{i}\theta_2} = -q_{2k-2}(\cdot + x_2)\,\mathrm{e}^{\mathrm{i}\theta_2}.$$

Hence, it is the case that for $\epsilon > 0$, $\lambda = 0$ is an eigenvalue of multiplicity 8.

4. Conclusions

In this paper, we have presented a systematic methodology for addressing the linear stability of nonlinear waves in perturbed Hamiltonian systems. Using as a starting point the full knowledge of the unperturbed problem and its symmetries, we can attack the perturbed problem, assessing its potential for persistence of nonlinear waves. The stability can be studied by means of a reduced matrix eigenvalue problem that yields leading-order information on the eigenvalues and eigenvectors of the perturbed problem. Also the Krein signature of the perturbed eigenvalues can be evaluated in this framework.

We have applied this technology to a system of interest to nonlinear optics applications, namely the coupled NLS equations, describing coupled optical fibres. For different solutions of the unperturbed limit, we have continued (in ϵ) the solutions numerically, and examined their full numerical linear stability. The latter results have been found in all the cases examined to be in very good agreement with the theoretical predictions. This agreement leads us to believe that this methodology could be a valuable tool in examining the stability of nonlinear waves in many other nonlinear wave bearing systems.

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Appendix. Spectrum of the stability matrix

The goal of this appendix is to determine the sign of the eigenvalues for a matrix which has the structure as that given in lemma 3.8. Let the vector $\mathbf{a} = (a_1, \ldots, a_n)^{\mathrm{T}} \in \mathbb{R}^n$, $n \geqslant 3$, be given, and assume that $a_i \neq 0$ for each $i = 1, \ldots, n$. Consider the symmetric matrix $A \in \mathbb{R}^{n\times n}$ which is given by

$$A_{i,i} = a_i + a_{i+1} \qquad A_{i,i+1} = -a_{i+1} \quad i = 1, \ldots, n$$

where the notation $A_{n,n+1} = A_{n,1}$ and $a_{n+1} = a_1$ is being used. Note that $\mathbf{v} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \ker(A)$.

First suppose that $a_j > 0$ for all j. It is easily seen that

$$\mathbf{x}^{\mathrm{T}} A\mathbf{x} = \sum_{i=1}^{n} a_i (x_i - x_{i-1})^2 \qquad x_0 = x_n.$$

If one sets $S = \mathrm{Span}((1, 1, \ldots, 1)^{\mathrm{T}})$, it is clear that $\mathbf{x}^{\mathrm{T}} A\mathbf{x} > 0$ for $\mathbf{x} \in S^{\perp}$; hence, all of the other eigenvalues of A are positive. Clearly, if $a_j < 0$ for all j, then A is negative definite on $S^{\perp}$.

Now suppose that $a_j < 0$ for some, but not all, value(s) of j. It is clear that A is now indefinite, so that it possesses at least one negative eigenvalue. One can write $A = B + C$, where both B and C are symmetric with

$$C_{1,1} = C_{n,n} = -C_{1,n} = -C_{n,1} = a_1$$

and $C_{i,j} = 0$ otherwise. It can be easily checked that zero is a simple eigenvalue of B with associated eigenvector $\mathbf{v} = (1, 1, \ldots, 1)^{\mathrm{T}}$. The matrix B also fits the structure given in [19, section 5]. It can then be concluded that

$$\mathrm{n}(B) = \#(j \geqslant 2 : a_j < 0) \qquad \mathrm{p}(B) = \#(j \geqslant 2 : a_j > 0).$$

Let

$$M = \#(j \geqslant 2 : a_j < 0).$$

By supposition $M \geqslant 0$. Order the eigenvalues of B as

$$\lambda_1(B) \leqslant \lambda_2(B) \leqslant \cdots \leqslant \lambda_M(B) < 0 < \lambda_{M+2}(B) \leqslant \cdots \leqslant \lambda_n(B).$$

It is straightforward to check that C is a rank 1 matrix with the nonzero eigenvalue being $2a_1$. If one orders the eigenvalues of A, $\lambda_i(A)$, in increasing order, then as a consequence of Weyl's theorem [20, theorem 4.3.7] one has that if $a_1 < 0$, then

$$\lambda_1(A) \leqslant \lambda_1(B) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_{n-1}(B) \leqslant \lambda_n(A) \leqslant \lambda_n(B)$$

i.e., the eigenvalues are interlaced and monotone decreasing, whereas if $a_1 > 0$, then

$$\lambda_1(B) \leqslant \lambda_1(A) \leqslant \lambda_2(B) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_n(B) \leqslant \lambda_n(A)$$

i.e., the eigenvalues are interlaced and monotone increasing. The following lemma can now be stated.

Lemma A.1. *The matrix A is positive definite if $a_j > 0$ for all j, negative definite if $a_j < 0$ for all j and indefinite otherwise. Suppose that*

$$\sum_{i=1}^{n}\left(\prod_{j\neq i} a_j\right) \neq 0.$$

Then $\mathrm{z}(A) = 1$. *Furthermore, if $a_1 < 0$, then*

$$\#(j \geqslant 2 : a_j < 0) \leqslant \mathrm{n}(A) \leqslant 1 + \#(j \geqslant 2 : a_j < 0)$$

while if $a_1 > 0$, then

$$\#(j \geqslant 2 : a_j < 0) - 1 \leqslant \mathrm{n}(A) \leqslant \#(j \geqslant 2 : a_j < 0).$$

Proof. All that is left to prove is the condition relating to the multiplicity of the zero eigenvalue. Performing Gaussian elimination on the matrix A to form the upper triangular matrix A^u, it is clear that row n is the zero row; furthermore, a tedious induction argument yields that

$$A^u_{n-1,n-1} = C\sum_{i=1}^{n}\left(\prod_{j\neq i} a_j\right) \qquad C \neq 0.$$

The result now follows. □

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